

# Modelling Post-Peak Behavior of Soils

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**Introduction:** The transition from peak to residual stress states in soil plays a critical role in slope failures and has garnered significant attention from researchers. Various curves -including polylines, quadratic functions, and cubic splines - have been employed to simulate the strength degradation between these states. However, to the author's knowledge, none have adequately captured the complexity of the peak-residual transition process.

A novel approach was introduced by Chiang (2017) and further developed by Lo and Huang (2021), utilizing a classical curve known as the *Versoria*, also referred to as the *Witch of Agnesi*, first proposed by Grandi in the 1700s. This curve offers a promising framework for simulating post-peak soil strength behavior.

## 11.1 Post-Peak Stress-Displacement Relationship

The post-peak segment of the shear stress–displacement ( $\tau$ – $\Delta$ ) curve is modeled using the *Versoria* (or *Witch of Agnesi*) curve, schematically illustrated in **Fig. 11.1.1**. For efficient integration into the overall stress–displacement framework, a normalized local coordinate system (X–Y) is adopted:

- The X-axis origin ( $X = 0$ ) aligns with the peak stress point ( $\tau_f$ ).
- The Y-axis origin ( $Y = 0$ ) corresponds to the asymptotic value of the residual stress ( $\tau_r$ ).

A distinctive feature of this curve is that the residual state ( $\tau_r$ ) is theoretically reached at an infinite displacement ( $\Delta$ ). In practice, however, a finite displacement ( $\Delta_r$ ), observed experimentally at the onset of the residual state (defined at  $X = 1$ ), can be used to approximate  $\tau_r$ . The shear strength in the post-peak regime ( $\tau_{post-peak}$ ) can be expressed as:

$$\tau_{post-peak} = \tau_f - (t - Y) \cdot \tau_f = \tau_f \cdot (1 - t + Y) \quad (11 - 1 - 1)$$

$$Y = \frac{t^3}{t^2 + X^2} \quad (11 - 1 - 2)$$

$$X = \frac{\Delta - \Delta_f}{\Delta_r - \Delta_f} \quad (11 - 1 - 3)$$

$$t = \frac{\tau_f - \tau_r}{\tau_f} \quad (11 - 1 - 4)$$

$$\Delta_r = \Delta_f \cdot \Delta_{ratio} \quad (11 - 1 - 5)$$

$t$ : normalized strength reduction between peak and residual states

$Y$ : normalized post-peak shear stress

$X$ : normalized post-peak shear displacement

$\Delta_f$ : shear displacement at peak stress state

$\Delta_r$ : shear displacement at the entrance of residual state

$\Delta_{ratio}$ : residual-to-peak displacement ratio

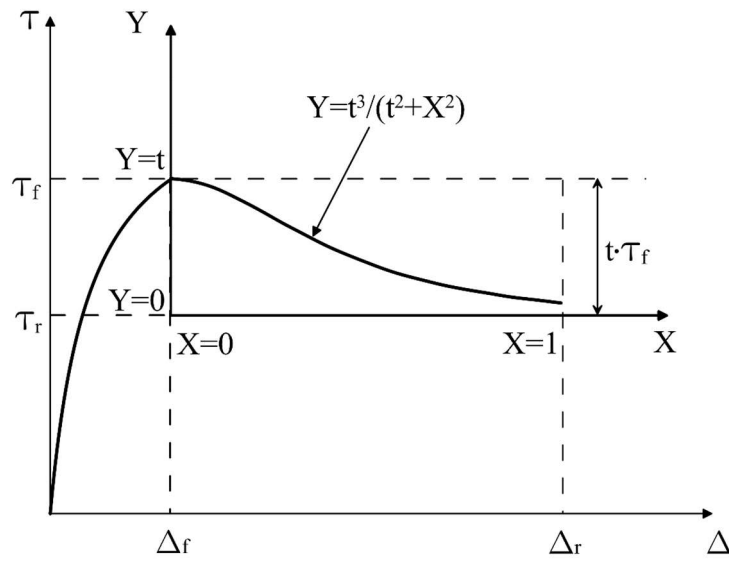


Fig. 11.1.1 A transformed coordinate system of the ‘Versoria’ curve for post-peak stress-displacement relationships

Based on the assumption of a constant friction angle ( $\varphi_r$ ) and the cohesion intercept,  $c_r=0$  at residual states,

$$\tau_r = \sigma'_n \cdot \tan \varphi_r \quad (15 - 1 - 6)$$

$\sigma'_n$ : effective normal stress

From **Eq. (11-1-4)**:

$$\tau_r = (1 - t) \cdot \tau_f \quad (11 - 1 - 7)$$

$$\tau_f = c_p + \sigma'_n \cdot \tan \varphi_p \quad (11 - 1 - 8)$$

$c_p$ ,  $\varphi_p$ : cohesion and internal friction angle, respectively, of soils at peak state

For a soil with known peak and residual strength parameters, the value of normalized strength reduction ( $t$ ) can be obtained from **Eqs. (11-1-7) and (11-1-8)** as:

$$t = 1 - \frac{\sigma'_n \cdot \tan \varphi_r}{c + \sigma'_n \cdot \tan \varphi_p} \quad (11 - 1 - 9)$$

For known values of  $\Delta$  and  $t$ , the normalized parameter  $X$  and  $Y$  can be calculated using **Eq. (11-1-3)** and **Eq. (11-1-2)**, respectively. Therefore, the corresponding shear stress ( $\tau$ ) can be calculated using **Eq. (11-1-1)** as:

$$\begin{aligned} \tau_{post-peak} &= \tau_f \cdot (1 - t + Y) \\ &= (c + \sigma'_n \cdot \tan \varphi_p) \cdot (1 - t + Y) \end{aligned} \quad (11 - 1 - 10)$$

Defining the post-peak strength of soil ( $\tau_{post-peak}$ ) according to the Mohr-Coulomb failure criterion:

$$\tau_{post-peak} = c_{post-peak} + \sigma'_n \cdot \tan \varphi_{post-peak} \quad (11 - 1 - 11)$$

Based on a well-known fact of  $c_{post-peak} = 0$  for soils under post-peak states, the available friction angle ( $\varphi_{post-peak}$ ) in the post-peak regime can be expressed as:

$$\begin{aligned} \varphi_{post-peak} &= \tan^{-1} \left( \frac{\tau_{post-peak}}{\sigma'_n} \right) \\ &= \tan^{-1} \left[ \frac{(c_p + \sigma'_n \cdot \tan \varphi_p) \cdot (1 - t + Y)}{\sigma'_n} \right] \end{aligned} \quad (11 - 1 - 12)$$

In the case of post-peak soil strength is considered in the slope displacement analysis using **SLOPE-ffdm 2.0**, **Eq. 11-1-12** is used to update the soil strength available along the slip surface when the condition of  $\Delta > \Delta_f$  is detected in the iterative calculations towards the convergence of slope displacement computations.

## 11.2 Post-Peak Soil Parameters

A series of medium-scale direct shear tests on various types of soil (Chu et al., 1989; Davoudi, 2011; Qui et al., 2000; Hsu, 2017; Wu, 2011; Ziaie Moayed et al., 2017) were analyzed to evaluate soil strength parameters in the peak–residual transitional state. Strength values at the peak and residual states, as obtained from the test results, are summarized in **Table 11.2.1**.

Table 11.2.1 Test conditions and hyperbolic soil strength parameters obtained in some medium to large scale direct shear test

Soil type (Reference)	Direct share boxes dimensions (mm)	$c_p$ (kPa)	$\phi_p$ (°)	$c_r$ (kPa )	$\phi_r$ (°)	USCS*
Lin-Kou gravelly soil (Chu et al., 1989)	1500*1500*750	0	49	0	39	GM
San-Yi gravelly soil (Chu et al., 1989)	1500*1500*750	0	56- 58	0	47- 48	GM
Toyoura sand, SP (Qiu et al., 2000)	300*300*300	0	38- 39	0	30- 32	SP
Gravelly lean clay (Davoudi, 2011)	500×500×300	11	41°	10	37	CL
Chi-Chi sand (Wu, 2011)	90*90*50	0	45- 49	0	42- 46	SW-SM
Mei-Shi sand (Hsu, 2017)	300*300*250	0	36- 53	0	34- 45	SP
Firouzkooch sand (Ziaie Moayed et al., 2017)	300*300*154	0	36	0	33	SP

\*USCS: Unified Soil Classified System

Experimental values of  $t$  and  $\Delta_{ratio}$  for seven test series are presented in **Table 11.2.2**. No clear trend was observed between  $t$  and confining stress ( $\sigma_n'$ ); in some test series,  $t$  increases with increasing  $\sigma_n'$ , while in others it decreases. In contrast, the tests consistently show a tendency for  $\Delta_{ratio}$  to decrease as  $\sigma_n'$  increases.

Given that the variations in  $t$  and  $\Delta_{ratio}$  across the investigated range of  $\sigma_n'$  are relatively minor in most of the tests, a simplified approach is proposed: applying stress-independent values of  $t$  and  $\Delta_{ratio}$  in practical implementations of the post-peak model.

Table 11.2.2 Experimental values of post-peak parameters

Soil type (Reference)	Confining stress, $\sigma_n'$ (kPa)	$t$	$\Delta_{ratio}$
Lin-Kou gravelly soil (Chu et al., 1989)	47	0.49	6.0
	138	0.41	6.0
	305	0.26	6.0
San-Yi gravelly soil (Chu et al., 1989)	75	0.31	5.7
	115	0.30	5.5
	130	0.30	5.4
Toyoura sand, SP (Qiu et al., 2000)	100	0.24	17
	150	0.26	11
	200	0.28	9
Gravelly lean clay (Davoudi, 2011)	20	0.1	3.0
	59	0.1	2.5
	98	0.11	2.0
Chi-Chi sand (Wu, 2011)	56	0.14	2
	109	0.12	1.8
	217	0.08	1.6
Mei-Shi sand (Hsu, 2017)	20	0.25	3.0
	50	0.25	2.7
	100	0.25	2.4
Firouzkooh sand (Ziaie Moayed et al., 2017)	109	0.21	4.0
	163	0.14	3.5
	218	0.06	3.0

### 11.3 Simulating Post-Peak Stress-Displacement Curve

Four test series were selected to demonstrate the effectiveness of the previously described post-peak stress-displacement model. These tests were chosen due to their distinct tendencies toward post-peak strength deterioration. As a first approximation - and to reduce the number of post-peak parameters required for practical application - the parameters  $t$  and  $\Delta_{ratio}$  were simplified by treating them as stress-independent. In **Table 11.3.1**, the values of  $t$  and  $\Delta_{ratio}$  represent mean values obtained through curve-fitting across tests with various confining stresses. An exception occurs in the case of  $\Delta_{ratio}$  for Toyoura sand. Curve-fitting produced values in the range of 9- 17; however, the value of  $\Delta_{ratio} = 4.0$  provides a better match to the experimental data, as shown in **Fig. 11.3.2**. Also note that SLOPE-ffdm 2.0 requires residual friction angle ( $\phi_r$ ) as an input parameter for implementing the post-peak model, instead of  $t$ . Therefore, a transformation using Eq. 11-1-9 is necessary:

$$\phi_r = \tan^{-1} \left[ \frac{(1 - t) \cdot (c + \sigma'_n \cdot \tan \phi_p)}{\sigma'_n} \right] \quad (11 - 3 - 1)$$

Values of  $\phi_r$  in Table 11-3-1 are obtained based on Eq. 11-3-1 under the condition of  $c = 0$ . For a cohesive soil with  $c > 0$ , a transformation from  $t$  into  $\phi_r$  may require an estimation of the average value of  $\sigma'_n$  on the potential failure surface.

Table 11.3.1 Peak and residual strength parameters used for verifying the post-peak stress-displacement model

Soil type (Reference)	Pre-peak model (Hyperbolic)					Post-peak model		
	$\phi_0$ (°)	$\Delta\phi$ (°)	$K$	$n$	$R_f$	$t$	$\phi_r$ (°)	$\Delta_{ratio}$
San-Yi gravelly soil (Chu et al., 1989)	58	6.5	620	1.0	0.83	0.3	48	6.0
Toyouura sand, SP (Qiu et al., 2000)	39	1.6	380	0.32	0.83	0.3	29	4.0
Chi-Chi sand (Wu, 2011)	49	7.8	500	0.7	0.83	0.13	45	2.0
Mei-Shi sand (Hsu, 2017)	53	12	580	0.9	0.83	0.25	45	2.5

Figures 11.3.1 through 11.3.4 illustrate this modeling approach: the hyperbolic model captures the pre-peak behavior, while the post-peak model describes the stress-displacement response beyond peak strength. Collectively, these figures demonstrate the effectiveness of the combined models across a range of soil types.

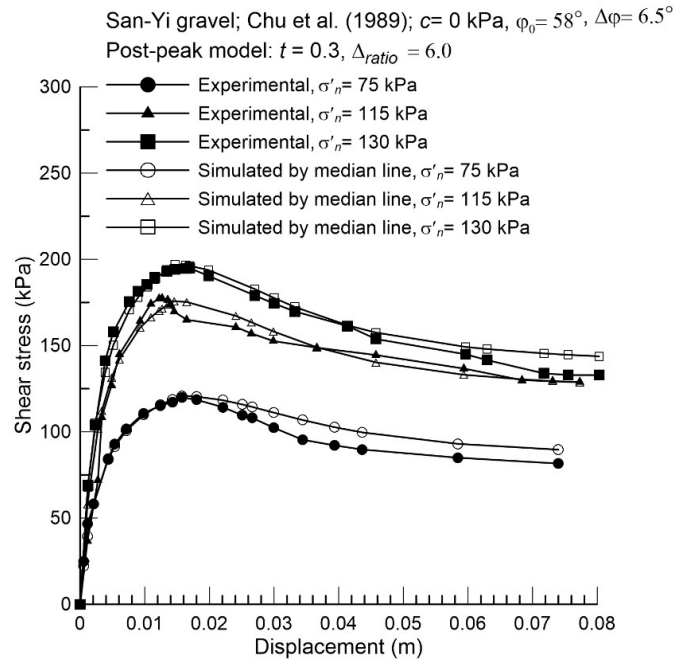


Figure 11.3.1 Comparisons of experimental and simulated stress-displacement curves for the San-Yi gravelly soil.

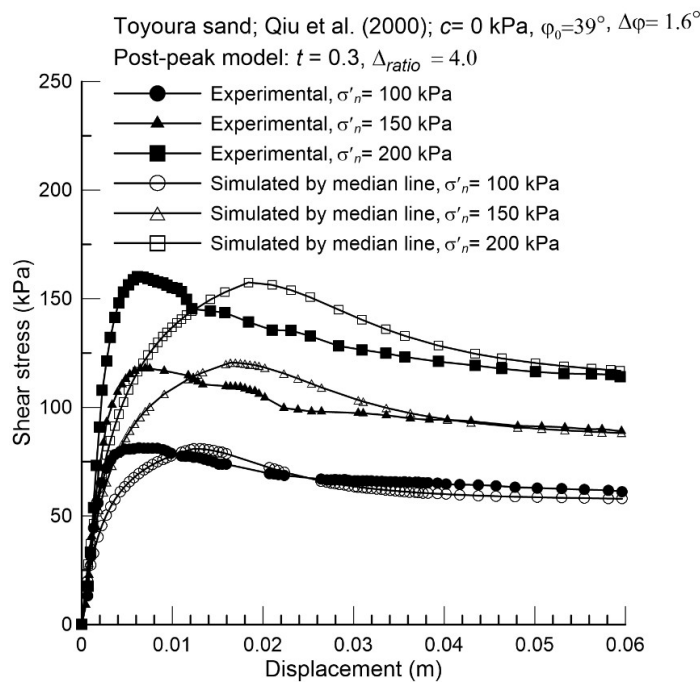


Figure 11.3.1 Comparisons of experimental and simulated stress-displacement curves for the Toyoura sand.

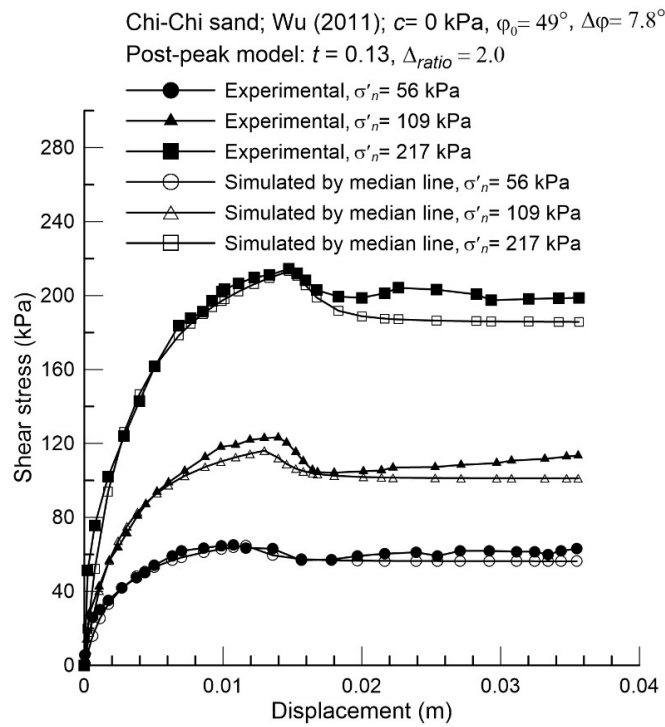


Figure 11.3.3 Comparisons of experimental and simulated stress-displacement curves for the Chi-Chi sand.

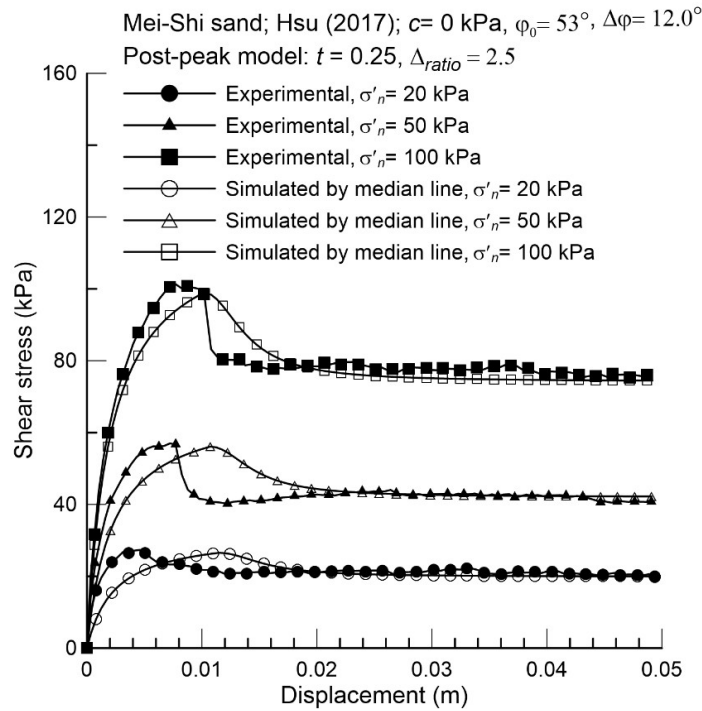


Figure 11.3.4 Comparisons of experimental and simulated stress-displacement curves for the Mei-Shi sand.

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